$\mathcal{PT}-$supersymmetry of singular harmonic oscillators

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Abstract

The Witten's supersymmetric interpretation of harmonic oscillator in one dimension ($D = 1$) is generalized to any $D > 1$. The emerging centrifugal barriers are regularized via a small imaginary shift of coordinates. The resulting $\mathcal{PT}$ symmetrized supersymmetry $sl(1/1)$ interrelates the spiked oscillator partners with different strengths of the barrier. The innovated form of the creation and annihilation operators is constructed.

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1 Introduction

Within the highly schematic formalism of the Witten's supersymmetric quantum mechanics [1], many difficult methodical problems related to the existence and spontaneous breakdown of the supersymmetric multiplets of bosons and fermions can be studied without any recourse to the less essential subtleties of the relativistic quantum field theory. The Witten's formalism recollects a years-long experience of mathematical physicists with the ordinary differential Schrödinger equation in one dimension [2]. It enables us to make a choice between the harmonic oscillator $V(x) = x^2$ and other candidates. The former, "linear" model is exceptional as it renders possible the construction of the fermionic *plus* bosonic Fock space (cf. review [3], p. 285).

A significant drawback of the Witten's methodical laboratory has been described by Jevicki and Rodrigues [4] as a partial or even complete failure of the recipe whenever the (super)potentials happen to be singular. We intend to re-consider critically the real depth of the latter difficulty, paying attention to the more dimensional harmonic oscillators which possess the "solvable" centrifugal singularity.

Following the recent idea of Bender and Boettcher [5] and working within their so called $\mathcal{PT}$ symmetric quantum mechanics [6, 7], we shall replace our more-dimensional (and, admittedly, spiked) harmonic oscillator Hamiltonians by the specific analytic continuation of their radial components [8]. In the other words, we shall work with the complex, pseudo-Hermitian Hamiltonians [9] and assume for simplicity that their spectrum remains real and discrete.

2 Complexification of coordinates

There hardly exists a textbook on quantum physics which would not mention harmonic oscillator. The shape of its potential can be used as a fairly realistic model in atomic physics and quantum chemistry. Within nuclear theory, the exact solvability

of the $A-$body harmonic oscillator looks almost like a miracle and proves particularly helpful. In field theory, the simplicity of the annihilation and creation operators

$$\mathbf{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \qquad \mathbf{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \tag{1}$$

enables us to understand free fields as "local" excitations $\Pi_\xi \, \mathbf{a}_\xi^\dagger \, |0\rangle$ of the vacuum.

In the recent studies of quantum fields with broken parity [10, 11], various Hamiltonians H have been considered and modified by the complex shift of coordinate,

$$q \to r = r(x) = x - i\varepsilon, \qquad x \in (-\infty, \infty). \tag{2}$$

The original symmetry with respect to the parity $\mathcal{P}$ was broken and replaced by a significantly weaker $\mathcal{PT}$ symmetry $[H, \mathcal{PT}] = 0$ where $\mathcal{T}$ mimicked the action of the time-reversal of fields. The Hermiticity of H was lost.

Notably, once we work with a "quasi-parity" quantum number $\beta = \pm 1/2$, the usual spectrum of the Hermitian harmonic oscillator Hamiltonian remains the same for all its shifted, non-Hermitian descendants [5],

$$E = E_n^{(\beta)} = 4n + 2\beta + 2, \qquad n = 0, 1, \ldots \tag{3}$$

The preservation of the discrete, real and semi-bounded form of these energies can be attributed to the $\mathcal{PT}$ symmetry of the underlying Hamiltonian [6]. Many exactly solvable $\mathcal{PT}$ symmetric models seem to support such a hypothesis [11, 12].

Once we fix the shift $\varepsilon > 0$ in eq. (2), the one-dimensional harmonic-oscillator Schrödinger equation can be replaced by a family of its singular generalizations

$$-\frac{d^2}{dr^2}\mathcal{L}(r) + \frac{\alpha^2 - 1/4}{r^2}\mathcal{L}(r) + r^2\,\mathcal{L}(r) = E\mathcal{L}(r) \tag{4}$$

with the same boundary conditions. The parameter

$$\alpha = (D-2)/2 + \ell + \omega > 0$$

varies with the spatial dimension D and with the non-negative integer angular momentum quantum number $\ell = 0, 1, \ldots$.

Our previous energy formula (3) remains valid and continuous at $\alpha = 1/2$ provided only that we put $\beta = \pm\alpha$ [13]. Off the point $\alpha = 1/2$ the spectrum of our new equation $H^{(\alpha)}\psi = E\psi$ is richer than in the Hermitian case. The analytic, confluent hypergeometric origin of equation (4) is to be seen in a new perspective. The ordinary Sturm-Liouville theory must be adapted to the new situation [14]. The norms have to be replaced by the pseudo-norms [15]. At the same time, our differential Schrödinger equation (4) stays *safely regularized* at any $\alpha > 0$.

In the generic case with $\alpha \neq 1/2$, our $\mathcal{PT}$ regularization ($\varepsilon \neq 0$) is only removable via the limiting transition $\varepsilon \to 0$ accompanied by a halving of the axis of coordinates. This means that we have to replace $r(x) = x - i\varepsilon$ by the radial and real $r \in (0, \infty)$. We must cross out all the states with $\beta < 0$. The $\varepsilon \to 0$ return to the Hermitian cases $D = 1$ or $D = 3$ remains transparent. A few details are mentioned in Table 1. The weird difference between the linear and spherical Hermitian oscillators can be attributed to the mere physically well motivated difference in boundary conditions in the origin.

This initiated our study. Its decisive encouragement came from the discovery of confluence of the different energy levels at certain couplings α. This anomaly remained unnoticed in ref. [8] and proves closely connected to the (*unavoided*) crossing of the energy levels at certain strengths of the barrier [13]. One can easily verify that the latter puzzling phenomenon takes place at all the integers α, i.e., at all the even dimensions $D = 2, 4, \ldots$ for the vanishing residual $\omega = 0$.

Such an observation is slightly discomforting. In the language of linear algebra, the exceptional character of the integers α can be intuitively explained via the occurrence of Jordan blocks in the canonical form of the non-Hermitian Hamiltonian matrix [16]. In order to simplify the discussion, we shall skip this point completely and, everywhere in what follows, we shall assume that $\alpha \neq 0, 1, 2, \ldots$.

3 Supersymmetrization of pairs of Hamiltonians

Let us first introduce a compact notation and recollect the essence of the Witten's supersymmetric quantum mechanics. It introduces the auxiliary "conjugate" operators $A = \partial_x + W$ and $B = -\partial_x + W$ in terms of the superpotential W. This factorizes certain Hamiltonians in full accordance with the old Schrödinger's recipe [2]. For the two partner operators

$$H_{(L)} = \hat{p}^2 + W^2 - W', \qquad H_{(R)} = \hat{p}^2 + W^2 + W' \tag{5}$$

it is easy to confirm that $H_{(L)} = B \cdot A$ while $H_{(R)} = A \cdot B$. In the Witten's language, the related two by two super-Hamiltonian and the two supercharges,

$$\mathcal{H} = \left[\begin{array}{cc} H_{(L)} & 0 \\ 0 & H_{(R)} \end{array} \right], \qquad \mathcal{Q} = \left[\begin{array}{cc} 0 & 0 \\ A & 0 \end{array} \right], \qquad \tilde{\mathcal{Q}} = \left[\begin{array}{cc} 0 & B \\ 0 & 0 \end{array} \right]$$

generate a representation of the superalgebra sl(1/1). The validity of the mixed anticommutation and commutation relations

$$\{\mathcal{Q}, \tilde{\mathcal{Q}}\} = \mathcal{H}, \qquad \{\mathcal{Q}, \mathcal{Q}\} = \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0, \qquad [\mathcal{H}, \mathcal{Q}] = [\mathcal{H}, \tilde{\mathcal{Q}}] = 0$$

mimics a supersymmetry between the bosonic and fermionic sectors of the Hilbert space. As already mentioned, the most appealing Fock-space picture of this type of the fermion – boson correspondence can be obtained in the "linear model" using the Hermitian one-dimensional harmonic oscillators. The scheme collapses after one moves to any dimension $D > 1$ {cf. the counterexample (488) in review [3]}.

Our $\varepsilon \neq 0$ regularization of Schrödinger equations makes the spiked harmonic oscillators eligible for supersymmetric treatment. Let us return to our equation (4) and to its well known solvability in terms of Laguerre polynomials,

$$\mathcal{L}(r) = \mathcal{L}_n^{(\beta)}(r) = \frac{n!}{\Gamma(n+\beta+1)} \cdot r^{\beta+1/2} \exp(-r^2/2) \cdot L_n^{(\beta)}(r^2), \qquad \beta = \pm\alpha\ . \tag{6}$$

A few *algebraic* consequences follow from the analytic $\mathcal{PT}$ regularity at $\varepsilon \neq 0$. We may pick up any real parameter $\gamma \neq 0, \pm 1, \pm 2, \ldots$ and construct the superpotential

$$W^{(\gamma)}(r) = -\frac{\partial_r \mathcal{L}_0^{(\gamma)}(r)}{\mathcal{L}_0^{(\gamma)}(r)} = r - \frac{\gamma + 1/2}{r}\,. \tag{7}$$

The supersymmetric recipe gives the $\gamma-$numbered partner Hamiltonians (5). After the appropriate insertions they may be given the explicit and compact harmonic oscillator form,

$$H_{(L)}^{(\gamma)} = H^{(\alpha)} - 2\gamma - 2, \qquad H_{(R)}^{(\gamma)} = H^{(\tilde{\alpha})} - 2\gamma, \qquad \alpha = |\gamma|, \qquad \tilde{\alpha} = |\gamma + 1|\,. \tag{8}$$

At all the non-integer $\gamma \neq 1/2$ the direct computations reveal that the superscripted operators

$$A^{(\gamma)} = \partial_r + W^{(\gamma)}, \qquad B^{(\gamma)} = -\partial_r + W^{(\gamma)}, \qquad \gamma \neq 0, \pm 1, \ldots \tag{9}$$

act on our (normalized, spiked and $\mathcal{PT}-$symmetrized) harmonic oscillator states in the transparent manner,

$$\begin{array}{ll} A^{(\gamma)}\,\mathcal{L}_{n+1}^{(\gamma)} = c_1(n,\gamma)\,\mathcal{L}_n^{(\gamma+1)}, & c_1(n,\gamma) = -2\sqrt{n+1}; \\ B^{(\gamma)}\,\mathcal{L}_n^{(\gamma+1)} = c_2(n,\gamma)\,\mathcal{L}_{n+1}^{(\gamma)}, & c_2(n,\gamma) = -2\sqrt{n+1}; \\ A^{(\gamma)}\,\mathcal{L}_n^{(-\gamma)} = c_3(n,\gamma)\,\mathcal{L}_n^{(-\gamma-1)}, & c_3(n,\gamma) = 2\sqrt{n-\gamma}; \\ B^{(\gamma)}\,\mathcal{L}_n^{(-\gamma-1)} = c_4(n,\gamma)\,\mathcal{L}_n^{(-\gamma)} & c_4(n,\gamma) = 2\sqrt{n-\gamma}. \end{array} \tag{10}$$

The former two rules were sufficient to define the well known one-dimensional annihilation and creation at $\alpha = 1/2$. The latter two lines have to be added in order to move us to any $\alpha \neq 1/2$. Our operators mediate the supersymmetric mapping and import an explicit $\gamma-$dependence in c_3 and c_4. Its definitely most embarrasing consequence is the re-curring singularity at $\gamma =$integer. This reflects the above-mentionend (unavoided) crossing of the energy levels.

4 Transitions between the different α

We observe that each of the two partner Hamiltonians generates the two quasi-parity subsets of the equidistant energy values. At each $n = 0, 1, \ldots$, the quadruplet of energies

$$E_{(L)}^{(+)} = 4n, \qquad E_{(L)}^{(-)} = 4n - 4\gamma, \qquad E_{(R)}^{(+)} = 4n + 4, \qquad E_{(R)}^{(-)} = 4n - 4\gamma \tag{11}$$

corresponds to the respective wave functions

$$\mathcal{L}_n^{(\gamma)}, \qquad \mathcal{L}_n^{(-\gamma)}, \qquad \mathcal{L}_n^{(\gamma+1)}, \qquad \mathcal{L}_n^{(-\gamma-1)} \;.$$

The ordering of these states depends on the sign of the parameter γ in a way summarized in Table 2. In this scheme, a mere shift of the energy spectrum (i.e., a coincidence of the Hamiltonians with α and $\tilde{\alpha}$) is, obviously, exceptional. The related parameter γ_e is given by the algebraic equation $|\gamma_e| = |\gamma_e + 1|$ with the unique solution $\gamma_e = -1/2$. In the Hermitian limit $\varepsilon \to 0$ we reproduce the one-dimensional example. Its supersymmetry is unbroken.

A few nontrivial $\alpha \neq 1/2$ examples of our generalized supersymmetric partnership are displayed in the pairs of the neighboring columns in Table 3. Its part (a) samples the two-way correspondence between the two different Hamiltonians $H^{(1/2)}$ and $H^{(3/2)}$. The $\gamma = -3/2$ $\mathcal{PT}$ supersymmetry between $H_{(L)} = H^{(3/2)} + 1$ and $H_{(R)} = H^{(1/2)} + 3$ is followed by the $\gamma = 1/2$ correspondence between the doublet $H_{(\tilde{L})} = H^{(1/2)} - 3$ and $H_{(\tilde{R})} = H^{(3/2)} - 1$. As a net result we obtain the appropriate generalization of the creation/annihilation pattern for the $p-$wave column of Table 1.

At $\gamma = -1/2$ we encounter the "degenerate" (and, in the present context, utterly exceptional) textbook $\gamma = -1/2$ pattern

$$\mathbf{a} \cdot \mathcal{L}_{n-1}^{(1/2)}(x) = \sqrt{2n-1}\,\mathcal{L}_{n-1}^{(-1/2)}(x), \qquad \mathbf{a} \cdot \mathcal{L}_{n}^{(-1/2)}(x) = -\sqrt{2n}\,\mathcal{L}_{n-1}^{(1/2)}(x)$$

$$\mathbf{a}^{\dagger}\,\mathcal{L}_{n-1}^{(-1/2)}(x) = \sqrt{2n-1}\,\mathcal{L}_{n-1}^{(1/2)}(x), \qquad \mathbf{a}^{\dagger}\,\mathcal{L}_{n-1}^{(1/2)}(x) = -\sqrt{2n}\,\mathcal{L}_{n}^{(-1/2)}(x)$$

where $\mathbf{a} \sim A^{(-1/2)}$ and $\mathbf{a}^\dagger \sim B^{(-1/2)}$. In a slightly re-ordered form, Table 3 (a) offers an alternative. Via the non-Hermitian detour and limit $\varepsilon \to 0$, another explicit annihilation pattern is obtained for the same $s-$wave oscillator. The new two-step mapping starts from the Hamiltonian $H_{(L)} = H^{(1/2)} - 3$ and gives, firstly, its $\mathcal{PT}$ symmetrically regularized non-Hermitian supersymmetric partner $H_{(R)} = H^{(3/2)} - 1$ [cf the last two columns in Table 3 (a)]. In the subsequent step, the similar partnership of the re-shifted $H_{(\tilde{L})} = H^{(3/2)} + 1$ returns us to the original $H_{(\tilde{R})} = H^{(1/2)} + 3$ [note that the shift differs from the one used in the first two columns in Table 3 (a)].

5 Creation and annihilation operators

At any $\gamma \neq -1/2$, our new supersymmetric pattern differs quite significantly from the Hermitian one. Firstly, in a fairly unusual way, the states can easily appear at a negative value of the energy [cf. the $E^{(-)}_{(L)0} = E^{(-)}_{(R)0} = -2$ right-hand side sample in Table 3 (a) etc]. Secondly, in a way reflecting the same possibility, the usual "unmatching" (i.e., absence of the $_{(R)}-$subscripted partner) at the vanishing energy $E^{(\beta)}_{(L)} = 0$ can occur for the excited states [cf., e.g., $\mathcal{L}^{(3/2)}_0 \to 0$ in Table 3 (b); the existence of such an anomalous "excited vacuum" is particularly important for the non-equidistant spectra]. Thirdly, in a sharp contrast to the Hermitian models, the lowest state exists (i.e., remains normalizable) in both the spectra of $H_{(L)}$ and $H_{(R)}$ [cf., e.g., the $\gamma = 3/2$ and $E = -6$ example $\mathcal{L}^{(-3/2)}_0 \to \mathcal{L}^{(-5/2)}_0$ in Table 3 (b)].

What can we expect from moving to the larger semi-integers $\alpha = |\gamma|$? Just a strengthening of the tendencies which were revealed in Table 3. Their features can be simply extrapolated. Thus, the large$-\gamma$ modifications of Table 3 will contain more lines at the bottom. For example, the $\gamma = 5/2$ supersymmetry between $H_{(L)} = H^{(5/2)} - 7$ and $H_{(R)} = H^{(7/2)} - 5$ will induce a ground-state mapping $\mathcal{L}^{(-5/2)}_0 \xrightarrow{A^{(5/2)}} \mathcal{L}^{(-7/2)}_0$. It appears at $E_{(L/R)} = -10$, witnessing just a continuing downward shift of the levels with the negative and decreasing superscripts.

Very similar conclusions can be drawn from the results generated at any real superscript value γ. We are near a climax of our study. A nice and transparent structure emerges form all the above $\mathcal{PT}$ supersymmetric assignments. Having spotted the difference between the regular and quasi-singular supersymmetries with $\alpha = 1/2$ and $\alpha \neq 1/2$, respectively, we are ready to study more deeply the non-equidistant spectra corresponding to the non-integer values of 2α.

Unless we reach the points of degeneracy α =integer, all our formulae remain applicable. The annihilation operators and their creation partners acquire the factorized, second-order differential form

$$A^{(-\gamma-1)} \cdot A^{(\gamma)} = A^{(\gamma-1)} \cdot A^{(-\gamma)} = \mathbf{A}(\alpha),$$

$$B^{(-\gamma)} \cdot B^{(\gamma-1)} = B^{(\gamma)} \cdot B^{(-\gamma-1)} = \mathbf{A}^{\dagger}(\alpha).$$

At any $\alpha \neq 0, 1, 2, \ldots$, they enable us to move along the spectrum of any harmonic oscillator Hamiltonian $H^{(\alpha)}$. We get

$$\mathbf{A}(\alpha) \cdot \mathcal{L}_{n+1}^{(\beta)} = c_5(n, \beta)\, \mathcal{L}_n^{(\beta)},$$

$$\mathbf{A}^{\dagger}(\alpha) \cdot \mathcal{L}_n^{(\beta)} = c_5(n, \beta)\, \mathcal{L}_{n+1}^{(\beta)},$$

$$c_5(n, \beta) = -4\sqrt{(n+1)(n+\beta+1)}, \qquad \beta = \pm\alpha.$$

This action is elementary and transparent.

6 Summary

We achieved a unified description of the spiked harmonic oscillators $H^{(\alpha)}$ within the $\mathcal{PT}$ symetric framework.

- The general $\mathcal{PT}$ supersymmetric partnership has been shown mediated by the "shape-invariance" operators $A^{(\gamma)}$ and $B^{(\gamma)}$.

- At any non-integer $\alpha > 0$ the role of the general creation and annihilation operators *for a given, single* Hamiltonian $H^{(\alpha)}$ has been shown played by their $\alpha-$dependent and $\beta-$preserving products $\mathbf{A}^{\dagger}(\alpha)$ and $\mathbf{A}(\alpha)$, respectively.

The only case where, up to a constant shift, the $\mathcal{PT}$ supersymmetric partners coincide corresponds to the case where the poles in $A^{(\gamma)}$ and $B^{(\gamma)}$ vanish. This is the only case tractable *without* the use of the $\mathcal{PT}$ regularization. In this sense, the traditional creation and annihilation using $\mathbf{a}^{\dagger} = B(-1/2)/\sqrt{2}$ and $\mathbf{a} = A(-1/2)/\sqrt{2}$ is slightly misleading since these operators *change* the quasi-parity β to $-\beta$. Such a correpsondence is not transferrable to any non-equidistant spectrum with $\alpha \neq -1/2$.

Our "natural" operators of creation $\mathbf{A}^{\dagger}(\alpha)$ and annihilation $\mathbf{A}(\alpha)$ are smooth near $\alpha = 1/2$. Their marginal (though practically relevant) merit lies in their reducibility to their regular (and hence, of course, state-dependent) first-order differential representation

$$\mathbf{A}(\alpha) \cdot \mathcal{L}_n^{(\beta)} = (2r\partial_r + 2\,r^2 - 4n - 2\beta - 1) \cdot \mathcal{L}_n^{(\beta)},$$

$$\mathbf{A}^{\dagger}(\alpha) \cdot \mathcal{L}_n^{(\beta)} = (-2r\partial_r + 2\,r^2 - 4n - 2\beta - 3) \cdot \mathcal{L}_n^{(\beta)}.$$

Let us notice that the change of variables $r \to y$ giving a simpler differentiation $2r\partial_r \to \partial_y$ reads $r = \exp 2y$ and would result in the so called Morse form of the Hamiltonian. This could, in principle, indicate that the Morse potentials with $\mathcal{PT}$ symmetry [17] would deserve more attention.

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Tables

Table 1.

A "disappearance" of wave functions in the Hermitian limit $\varepsilon \to 0$ at $D = 3$.

harmonic oscillator			energy
linear ($D = 1$)	spherical ($D = 3$)		
	$s -$ wave	$p -$ wave	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\|4\rangle = \mathcal{L}_2^{(-1/2)}$	$-$	$\|1\rangle = \sqrt{2}\mathcal{L}_2^{(3/2)}$	9
$\|3\rangle = \mathcal{L}_1^{(1/2)}$	$\|1\rangle = \sqrt{2}\mathcal{L}_1^{(1/2)}$	$-$	7
$\|2\rangle = \mathcal{L}_1^{(-1/2)}$	$-$	$\|0\rangle = \sqrt{2}\mathcal{L}_1^{(3/2)}$	5
$\|1\rangle = \mathcal{L}_0^{(1/2)}$	$\|0\rangle = \sqrt{2}\mathcal{L}_0^{(1/2)}$	$-$	3
$\|0\rangle = \mathcal{L}_0^{(-1/2)}$	$-$	$-$	1

Table 2.

Supersymmetry of harmonic oscillators at non-integer $\alpha = |\gamma|$. Superpotential (7) gives the partner Hamiltonians (5) and quadruplets of energies $E_a \le E_b \le E_c \le E_d$ at each $n = 0, 1, \ldots$.

range of α	$(0, \infty)$	$(0, 1)$	$(1, \infty)$
parameters			
$\tilde{\alpha} = \|\gamma + 1\|$	$\alpha + 1$	$1 - \alpha$	$\alpha - 1$
γ	α	$-\alpha$	$-\alpha$
Hamiltonians			
$H^{(\gamma)}_{(L)}$	$H^{(\alpha)} - 2\tilde{\alpha}$	$H^{(\alpha)} - 2\tilde{\alpha}$	$H^{(\alpha)} + 2\tilde{\alpha}$
$H^{(\gamma)}_{(R)}$	$H^{(\tilde{\alpha})} - 2\alpha$	$H^{(\tilde{\alpha})} + 2\alpha$	$H^{(\tilde{\alpha})} + 2\alpha$
energies			
$E_d = E_{(R)} \left[\text{of } \mathcal{L}_n^{(\tilde{\alpha})}\right]$	$4n + 4$	$4n + 4$	$4n + 4\alpha$
$E_c = E_{(L)} \left[\text{of } \mathcal{L}_n^{(\alpha)}\right]$	$4n$	$4n + 4\alpha$	$4n + 4\alpha$
$E_b = E_{(R)} \left[\text{of } \mathcal{L}_n^{(-\tilde{\alpha})}\right]$	$4n - 4\alpha$	$4n + 4\alpha$	$4n + 4$
$E_a = E_{(L)} \left[\text{of } \mathcal{L}_n^{(-\alpha)}\right]$	$4n - 4\alpha$	$4n$	$4n$

Table 3.

Supersymetry for the singular Hamiltonian $H^{(3/2)} = p^2 + (x - i\varepsilon)^2 + 2/(x - i\varepsilon)^2$.

(a) $\gamma = -3/2$ and $-\gamma - 1 = 1/2$.

$E_{(L/R)}$	$\vert n_{(L)}\rangle$	$\xrightarrow{A^{(-3/2)}}$	$\vert n_{(R)}\rangle = \vert n_{(\tilde{L})}\rangle$	$\xrightarrow{A^{(1/2)}}$	$\vert n_{(\tilde{R})}\rangle$	$E_{(\tilde{L}/\tilde{R})}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
14	$\mathcal{L}_2^{(3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(1/2)}$	$\rightarrow$	$\mathcal{L}_1^{(3/2)}$	8
12	$\mathcal{L}_3^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-3/2)}$	6
10	$\mathcal{L}_1^{(3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(1/2)}$	$\rightarrow$	$\mathcal{L}_0^{(3/2)}$	4
8	$\mathcal{L}_2^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-3/2)}$	2
6	$\mathcal{L}_0^{(3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(1/2)}$	$\rightarrow$	0	0
4	$\mathcal{L}_1^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-3/2)}$	-2
2	$-$		$-$		$-$	-4
0	$\mathcal{L}_0^{(-3/2)}$	$\rightarrow$	0	$\rightarrow$	$-$	-6

(b) $\gamma = 3/2$ and $-\gamma - 1 = -5/2$.

$E_{(L/R)}$	$\vert n_{(L)}\rangle$	$\xrightarrow{A^{(3/2)}}$	$\vert n_{(R)}\rangle = \vert n_{(\tilde{L})}\rangle$	$\xrightarrow{A^{(-5/2)}}$	$\vert n_{(\tilde{R})}\rangle$	$E_{(\tilde{L}/\tilde{R})}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
8	$\mathcal{L}_2^{(3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(5/2)}$	$\rightarrow$	$\mathcal{L}_1^{(3/2)}$	14
6	$\mathcal{L}_3^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_3^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-3/2)}$	12
4	$\mathcal{L}_1^{(3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(5/2)}$	$\rightarrow$	$\mathcal{L}_0^{(3/2)}$	10
2	$\mathcal{L}_2^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-3/2)}$	8
0	$\mathcal{L}_0^{(3/2)}$	$\rightarrow$	0	$\rightarrow$	$-$	6
-2	$\mathcal{L}_1^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-3/2)}$	4
-4	$-$		$-$		$-$	2
-6	$\mathcal{L}_0^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-5/2)}$	$\rightarrow$	0	0